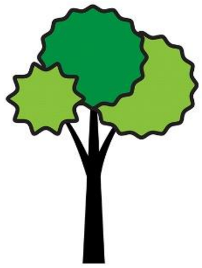
Tree Trunk Math



A tree, a large Douglas fir, had fallen down at the cabin. I saw it lying at the end of the lane as I drove up and was immediately thankful that it had not fallen across the road or onto the cabin itself. I enlisted the aid of a neighbor, Frank, and with his chainsaw, we limbed and trimmed until we had a mostly bare trunk lying before us. Twenty inches across at the base, it tapered to a spindly top that had broken off in the fall. I paced along its length, estimating about 2 and 1/2 feet per stride, and found that the tree had been a little over 100 feet tall! We worked for the better part of the next day to get it cut up for firewood. Along the way I asked Frank to cut me a "tree cookie," a slice about two inches thick taken near the base of the trunk. I wasn't sure what I would do with it, but something would come up.

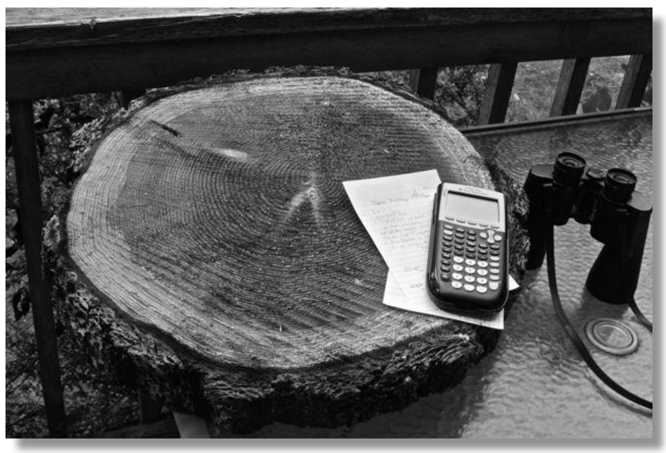


Figure 1. *The tree cookie*

Weeks later, there it was, sitting on a table on the deck where I had left it. Taking a closer look, I counted about 70 concentric rings, indicating as many years of growth. The rings told an interesting story. That tree must have sprouted sometime around 1950. Harry Truman was president. TVs were black and white, and a loaf of bread cost 12 cents. Rings close together meant that not much growth had taken place in that year. Fat rings meant good years for growing. But clearly the farther out you went from the center, the closer the rings were to one another. Did growth slow down as the tree aged? Thinking further I realized that although the rings were closer together in the later years, the tree at that time was taller, so the overall volume and therefore mass or weight added to the trunk each year might be the same. It might even be greater—or less. This needed investigation.

I found a tape measure and a pad of paper. I counted out 5 rings from the center and measured the distance at 7/8 of an inch. That meant that the tree at 5 years old had been a little short of two inches in diameter. I could see, though, that the rings were not perfect circles and that I would get different measurements depending on which direction I headed out from the center. Perhaps the tree grew more vigorously on one side than another. What caused that? Was it the sun shining mostly on the south-facing side? Was the wind blowing predominantly from one direction or another? Would that make a difference? Saving those questions for later, I decided to measure out from the center in four different directions, each 90 degrees from the last, and to take the average of the four measurements as the approximate radius of the trunk at that time. I did this for 5, 10, 20, 30, 40, 45, and 70 years. Table 1 shows the measurements.

	5 yrs.	**10 yrs.**	**20 yrs.**	**30 yrs.**	**40 yrs.**	**45 yrs.**	**70 yrs.**
Radius 1	$\frac{7}{8}$	$1\frac{7}{8}$	$4\frac{1}{8}$	6	$7\frac{1}{4}$	$7\frac{3}{4}$	$10\frac{1}{4}$
Radius 2	$\frac{7}{8}$	$1\frac{3}{4}$	$3\frac{5}{8}$	$4\frac{11}{16}$	$5\frac{7}{8}$	$6\frac{1}{4}$	$9\frac{1}{4}$
Radius 3	$\frac{15}{16}$	$1\frac{13}{16}$	$3\frac{9}{16}$	5	$6\frac{1}{4}$	$6\frac{3}{4}$	$9\frac{1}{8}$
Radius 4	$1\frac{1}{8}$	$2\frac{5}{16}$	$4\frac{7}{8}$	7	$8\frac{1}{2}$	9	12
Conveniently rounded **average radius**	$\frac{15}{16}$	2	4	$5\frac{11}{16}$	7	$7\frac{1}{2}$	$10\frac{1}{4}$

Table 1. *Radius measurements and their approximate means in inches*

So, I concluded that at 5, 10, 20, 30, 40, 45, and 70 years, the trunk had an average radius of about $\frac{15}{16}$, 2, 4, $5\frac{11}{16}$, 7, $7\frac{1}{2}$ and $10\frac{1}{4}$ inches. Getting back to my question about how the tree had put on weight over the years, I thought about it this way: Each year, the tree added a new layer—a new coat over the old—and the tree was always thicker at the base and thinner at the top. I decided that I could approximate the shape of the trunk as a very tall cone. Each year, the trunk got a little taller and a little thicker, so the cone got a little bigger.

The volume of a cone is given by the expression $\frac{1}{3}\pi r^2 h$, where r is the radius of the cone at the base and h is the cone's height. At 5 years of age, for example, the radius was a little under an inch—but what was the height? There must be a way to estimate the height given the diameter.

Figure 2. *Tree cookie with rings for 10, 20, 30 and 40 years emphasized.*

I did some research. It turns out that the heights of trees, and Douglas fir trees in particular, depend on a good deal more than how long they have been growing. As you would expect, the height they attain will depend on their growing conditions. Also, it seems that they tend to grow more slowly for the first 5 years, then more rapidly for the next 50 or so years, and then return to a slower rate of growth. They can grow for more than 1,000 years and attain heights over 300 feet. From ArborDay.org's tree guide (http://www.arborday.org/trees/treeGuide/), I learned that the Douglas fir is considered a medium growth tree and that such a tree, all other things being equal, can be expected to grow between 13–24 inches per year. Ignoring the variance of the rate over the life of the tree, I now had a simple mathematical model that gave me a rough estimate of the tree's height for any given year. I took 18 inches per year to be the approximate middle of the growth range given by ArborDay.org. It was a convenient number, 1 and 1/2 feet per year. Now I could calculate an "*h*," a height, to go with my "*r*," the radius I had measured from the tree cookie. That meant that I could get an estimate of the volume of the tree—the trunk at least—for any year by using the formula for the volume of a cone. Then, with a little subtraction, I could calculate the change in volume that had taken place at different points in the tree's history.

Table 2 shows the estimates of the tree's height, measured radius, and thus the volume of the tree's trunk as a function of the tree's age.

Years	**5 yrs.**	**10 yrs.**	**20 yrs.**	**30 yrs.**	**40 yrs.**	**45 yrs.**	**70 yrs.**
Estimated height (feet)	7.5	15	30	45	60	67.5	105
Approx. radius (inches)	0.94	2.00	4.00	5.69	7.00	7.50	10.25
Approximate volume of trunk (cubic inches)	83	754	6,032	18,292	36,945	47,713	138,627

Table 2. *Estimated height, measured radius, and calculated volume of the tree trunk in cubic inches*

The results were a little surprising, and I checked them a couple of times. Could the tree trunk go from 0 to 83 cubic inches in the first 5 years and then gain about 670 additional cubic inches in the next 5 years? If it had a volume of about 750 in.^{3} after 10 years, then, growing at the same rate, it would have about 1,500 in.^{3} after 20 years—but my estimate put it at more than 6,000 in.^{3}. Wow! Further subtraction showed that the trunk had gained, in successive 10-year intervals, 754 in.^{3} (0–10 years), then 5,278 in.^{3} (10–20 years), then 12,260 in.^{3} (20—30 years), then 18,653 in.^{3} (30–40 years), and then 101,632 in.^{3} in the final 30 years. I went to my computer, started a spreadsheet, and made a chart of volume as a function of time (see figure 3). That was certainly not a straight line. The trunk was gaining volume at a rate that accelerated over time.

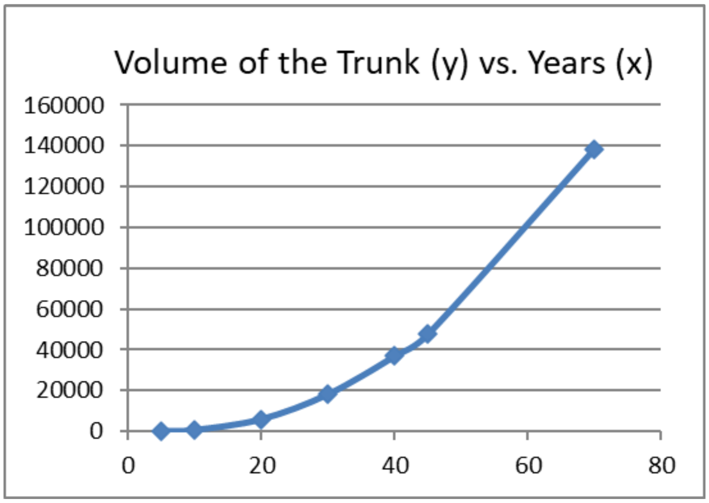


Figure 3. *Approximate volume of trunk vs years*

Giving this a little thought, I could see that it made sense. Trees produce glucose, their food, through photosynthesis. Photosynthesis happens in the leaves (needles in this case). The needles are on the branches. The taller the trunk, the more branches the tree has. The more branches, the more food, and so on, so it makes sense that the rate of growth would accelerate, at least for a while. Of course, that cannot go on forever. The mass of a tree this size is enormous. And if it continued to grow more and more quickly, wouldn't it eventually collapse under its own weight? How much *did* it weigh?

Knowing the volume of the trunk, I should be able to get a rough estimate if I knew how much each cubic inch weighed. I know that wood floats, so it is less dense than water; and I know that water weighs one gram per cubic centimeter. Guessing that the fir tree's wood, which is considered to be a "soft" wood, might be half as dense as water (not far off, actually; I looked it up later!) and knowing that there are 2.54 centimeters in every inch, I worked up a little conversion and, with the help of my calculator, found that 138,627 in.^{3}, the volume of the trunk at 70 years of age, is equivalent to 2,271,690 cm^3. If each cubic centimeter weighed half a gram, that made about 1,136 kg. Since a kilogram is about 2.2 pounds, that is about 2,500 lbs.—well over a ton! The trunk was about 100 feet tall. If we cut it into pieces that were 1.5 feet long, then split each of these into an average of four pieces, that would give about 265 pieces. Dividing 2,500 lbs. by 265 pieces told me that the average weight of my chunks of firewood would be about 9 and 1/2 pounds. (An average high school math text, by the way, comes in at about 3 or 4 pounds.)

Looking again at the steeply rising curve in my chart and remembering that the tallest Douglas firs were not much more than 300 feet tall, I wondered what happens in trees to slow down their growth over time. Do their cellular processes simply become less efficient after 100 years or so? (At 62 years of age, I feel that I am beginning to develop some personal understanding of this decline.)

Finally, I returned to the tree cookie. The rings *were* closer together as the tree got older, but the area in each successive ring, being dependent on the *square* of the radius, was increasing more rapidly as the tree grew older. The volume of the trunk, as modeled by an ever-taller cone, needed only a little change in radius to produce a big change in volume.

The tree rings did tell a story about the tree, and the story was made more interesting and more important to me with the aid of some simple mathematics. Looking up from the tree cookie, I could see dozens of other trees, still living and growing, and an infinite number of other instances throughout the forest calling out or waiting quietly for my attention, my appreciation, and, sometimes, my mathematics.

Lesson Plan

Learn more about implementing Tree Trunk Math in your classroom by exploring the Illuminations lesson here! Then, share your experiences using Math Sightings on social media with the hashtag #MathSightings.